QUANTIFIED MODAL LOGIC

This is a book on first-order modal logic. Adding quantifier machinery to classical propositional logic yields first-order classical logic, fully formed and ready to go. For modal logic, however, adding quantifiers is far from the end of the story, as we will soon see. But certainly adding quantifiers is the place to start. As it happens, even this step presents complications that do not arise classically. We say what some of these are after we get language syntax matters out of the way.

4.1. FIRST-ORDER FORMULAS

We have the same propositional connectives that we had in Chapter 1, and the same modal operators. We add to these two quantifiers, ∀ (for all, the universal quantifier) and ∃ (there is, the existential quantifier). Just as with the modal operators, these turn out to be interdefinable, so sometimes it will be convenient to take one as basic and the other as defined.

We assume we have available an infinite list of one place relation symbols, \( P_1^1, P_1^2, P_1^3, \ldots \), an infinite list of two place relation symbols, \( P_2^1, P_2^2, P_2^3, \ldots \), an infinite list of three place relation symbols, \( P_3^1, P_3^2, P_3^3, \ldots \), and so on. An \( n \)-place relation symbol is said to be of arity \( n \). We will generally be informal in our notation and use \( P, R \), or something similar for a relation symbol, with its arity determined from context. One place relation symbols are sometimes referred to as predicate symbols.

We also assume we have available an infinite list of variables, \( v_1, v_2, v_3, \ldots \). Here too we will generally be informal in our notation, and write \( x, y, z \), and the like, for variables.

DEFINITION 4.1.1. [Atomic Formula] An atomic formula is any expression of the form \( R(x_1, x_2, \ldots, x_n) \), where \( R \) is an \( n \)-place relation symbol and \( x_1, x_2, \ldots, x_n \) are variables.

The propositional letters of Chapter 1 were stand-ins for propositions, which were not further analyzed. In a first-order logic, atomic formulas have more structure to them. Think of a relation symbol as standing for a relation, such as is-the-brother-of (two-place) or is-a-number-between (three-place). An atomic formula like \( R(x, y) \) is supposed to mean: the relation that \( R \)
stands for holds of the things that are the values of \( x \) and \( y \). This will be given a precise meaning when semantics is discussed.

Next we define the notion of first-order formula, and simultaneously we define the notion of free variable occurrence. Anticipating notation that we are about to define, we will see that \((\forall x)P(x) \supset Q(x))\) is a formula. In it, the occurrence of \( x \) in \( P(x) \) is within the scope of the quantifier \((\forall x)\), and is not counted as free. What \( x \) is supposed to stand for makes no difference in the interpretation of \((\forall x)P(x)\). If we say \( x \) stands for the king of spades, or we say \( x \) stands for the king of rock and roll, it doesn’t affect the interpretation of \((\forall x)P(x)\) — indeed, we could have written \((\forall z)P(z)\) just as well. The formula simply says everything has property \( P \), and variables allow us to say this clearly. On the other hand, the occurrence of \( x \) in \( Q(x) \) is not within the scope of any quantifier, so what this \( x \) stands for can make a difference. Thus in the formula \((\forall x)P(x) \supset Q(x))\) only one occurrence of \( x \) is to be taken as free, and so is affected by the value assigned to \( x \). Now for the details.

**Definition 4.1.2. [First-Order Modal Formulas]** The sets of first-order formula and free variable occurrence are determined by the following rules.

1. Every atomic formula is a formula; every occurrence of a variable in an atomic formula is a free occurrence.
2. If \( X \) is a formula, so is \( \neg X \); the free variable occurrences of \( \neg X \) are those of \( X \).
3. If \( X \) and \( Y \) are formulas and \( \circ \) is a binary connective, \((X \circ Y)\) is a formula; the free variable occurrences of \((X \circ Y)\) are those of \( X \) together with those of \( Y \).
4. If \( X \) is a formula, so are \( X \) and \( \circ X \); the free variable occurrences of \( \Box X \) and of \( \circ X \) are those of \( X \).
5. If \( X \) is a formula and \( v \) is a variable, both \((\forall v)X\) and \((\exists v)X\) are formulas; the free variable occurrences of \((\forall v)X\) and of \((\exists v)X\) are those of \( X \), except for occurrences of \( v \).

Any variable occurrences in a formula that are not free are said to be bound.

**Example 4.1.3.** Consider \((\exists y)((\forall x)P(x, y) \supset \Box Q(x, y))\), where \( P \) and \( Q \) are both two-place relation symbols. We sketch why this is a formula, and determine which variable occurrences are free. To help make things clear, we display free variable occurrences in formulas using bold-face.

Both \( P(x, y) \) and \( Q(x, y) \) are atomic formulas, hence both are formulas, and all variable occurrences are free.

Since \( Q(x, y) \) is a formula, so is \( J Q(x, y) \), and it has the same free variable occurrences.

Since \( P(x, y) \) is a formula, so is \((\forall x)P(x, y)\), and the free variable occurrences are those of \( P(x, y) \), except for occurrences of \( x \), hence only the occurrence of \( y \) is free.

Now \((\forall x)P(x, y) \supset \Box Q(x, y))\) is a formula, with free variable occurrences as indicated.

Finally, \((\exists y)((\forall x)P(x, y) \supset \Box Q(x, y))\) is a formula, and its free variable occurrences are those of \((\forall x)P(x, y) \supset \Box Q(x, y))\), except for occurrences of \( y \). Thus finally, only one occurrence of \( x \) is free.

When models are defined later on, we will see that a formula without free variables is simply true at a world of a model, or not. But for a formula with free variables, additional information must be supplied before truth can be determined, namely values for the free variables. This suggests that formulas without free variables will play a special role.

**Definition 4.1.4. [Sentence]** We call a formula with no free variable occurrences a sentence or a closed formula.

Finally, we will continue our earlier practice of informally omitting outer parentheses from formulas, and using differently shaped parentheses.

**Exercises**

**Exercise 4.1.1.** Verify that the following are formulas, and determine which are the free variable occurrences. Also, determine which of the following are sentences.

1. \((\forall x)(\exists y)P(x, y) \supset (\exists y)\Box Q(x, y))\)
2. \((\exists x)(\Box P(x) \supset (\forall x)\Box P(x))\)
3. \((\forall x)((\exists y)R(x, y) \supset R(y, x))\)

**4.2. An Informal Introduction**

Let’s look at the kinds of things we can say with first-order formulas before quantifiers are taken into account.

Contrast the formula

\[ (4.1) \quad F(x) \]

with the formula

\[ (4.2) \quad \Box F(x). \]
(4.1) says that an object \( x \) has the property \( F \)—in the actual world, of course. (4.2) says that an object \( x \) has the property \( F \) necessarily, i.e., in every possible world.\(^{14}\) The \( x \) that occurs in both (4.1) and (4.2) is a free variable, not a name or a description or a function name of any sort. Our formal machinery will include a mechanism for assigning values—objects—to free variables. Informally, we will just refer to the object \( x \). It is important to keep in mind that it is the object itself—the value of \( x \)—that we are speaking about, and not the variable.

The contrast between (4.1) and (4.2) is just the contrast, for example, between saying that God exists and saying that God necessarily exists. If \( F \) stands for "exists" and \( x \) has God as its value, then (4.1) says that God exists in this world. It does not, of course, say that he exists in every possible world. And if he did not exist in every possible world, then his existence would only be contingent. This is a matter of considerable importance in classical theology. For, it has been claimed that God differs from other things in that it is part of God's essence that he exists. This is to say that it is an essential property of him that he exists. And, indeed, (4.2) can be read as \( F \) is an essential property of \( x \).

On the crudest version of the Ontological Argument, the claim that it is part of God's essence that he exists is interpreted as (4.2), and so the fact of his existence, (4.1), follows immediately via \( \square F(x) \supset F(x) \) (which is valid in a frame at least as strong as \( T \)).

But there are more subtle ways of understanding the claim that it is part of God's essence that he exists. Consider

\[
(4.3) \quad F(x) \supset \square F(x),
\]

which says that \( x \) has \( F \) necessarily if it has \( F \) at all. This is not generally true: just because \( x \) is \( F \) in the actual world, it does not follow that it is \( F \) in every possible world. But something special happens for the case of God's existence. (4.2) categorically asserts that \( x \) is \( F \) in every possible world; (4.3) asserts this only hypothetically, i.e., on the condition that \( x \) is \( F \) in this world. So, for the case of God's existence, the contrast is between saying that God necessarily exists and saying that God necessarily exists if he exists. The distinction is important. It has been a matter of some concern among Descartes' commentators which of these two is the proper conclusion of the version of the Ontological Argument he presents in Meditation V. For if it is (4.3), then, of course, God's existence has not been established.

We can make yet finer discriminations here if we contrast (4.3) with

\[
(4.4) \quad \square (F(x) \supset \square F(x)),
\]

\(^{14}\) Understood, of course, as "in every accessible possible world." We will generally suppress "accessible" in our informal discussion.

(4.4) contains one \( \square \) nested inside another. This is where informally interpreting modal formulas becomes difficult; it is also where possible world semantics is most helpful. (4.4) says that (4.3) is necessary, i.e., that it holds in every possible world. Now, (4.3) says that the following holds in the actual world:

Either \( x \) isn't \( F \) or is \( F \) in every possible world.

So, (4.4) says that this holds in any world.

Returning to our example, the contrast is between saying that God necessarily exists if he exists and saying that Necessarily, God necessarily exists if he exists. Once again, one might balk at the idea that it is only a contingent property of him that he necessarily exists if he exists.

Our first-order language enables us to make very subtle distinctions that greatly enrich our understanding of traditional modal claims. In this section, we have focussed on open sentences, i.e., sentences that contain free variables. In the next section, we will look at quantified sentences.

**Exercises**

**Exercise 4.2.1.** Consider the claim

It is necessary that an omniscient being is essentially omniscient

Does this require that an omniscient being exist in more than one possible world (if it exists at all)? Suppose it is possible that there is an omniscient being. Does it follow that one exists?

**4.3. Necessity De Re and De Dicto**

The sentence

\[
(4.5) \quad \text{Everything is necessarily } F,
\]

contains an ambiguous construction that has caused much confusion in the history of modal logic. We can express these two readings of (4.5) a bit more clearly as

\[
(4.6) \quad \text{It is a necessary truth that everything is } F.
\]

and

\[
(4.7) \quad \text{Each thing is such that it has } F \text{ necessarily.}
\]
The medieval logicians were aware of these two interpretations. (4.6) expresses that a proposition [dictum] is necessary, and so it is an example of what they called necessity de dicto. (4.7) expresses that a thing [res] has a property necessarily. It is an example of necessity de re.

We find the same de re de dicto ambiguity in

\[ (4.8) \quad \text{Something necessarily exists} \]

On the one hand, (4.8) could mean

\[ (4.9) \quad \text{It is necessarily true that something exists.} \]

This is the de dicto reading, and it is true in our semantics simply because, following standard procedure, quantifier domains are taken to be non-empty. On the other hand, the de re reading of (4.8),

\[ (4.10) \quad \text{Something has the property of existence essentially,} \]

is quite controversial. (4.10) commits us to a necessary existent—perhaps God, but, in any event, an object that exists in every possible world.

Here is an example familiar to those who have followed the recent philosophical literature on modal logic,

\[ (4.11) \quad \text{The number of planets is necessarily odd.} \]

On the de dicto reading, (4.11) says that a certain proposition is necessary, namely, the proposition that the number of planets is odd. And on this de dicto interpretation, (4.11) is clearly false, for surely it is only a contingent fact that the number of planets is odd. Had history been different, there could have been 4, or 10, or 2 planets in our solar system. But the de re reading of (4.11) says, of the number of planets, that that number is necessarily odd. And as a matter of fact, the number of planets is 9, and it is an essential property of the number 9 that it is odd.

It turns out that the de re de dicto distinction is readily expressible in our first-order modal language as a scope distinction. We take the de dicto (4.6) to be

\[ (4.12) \quad \Box (\forall x) F(x) \]

which asserts, of the statement \((\forall x) F(x)\), that it is necessary. And we take the de re (4.7) to be

\[ (4.13) \quad (\forall x) \Box F(x) \]

which asserts, of each thing, that it has \(F\) necessarily. (4.12) says, In every possible world, everything is \(F\). (4.13) says, Everything is, in every possible world, \(F\). We take the de dicto and de re readings of

Something is necessarily \(F\)

to be, respectively,

\[ (4.14) \quad \Box (\exists x) F(x). \]

and

\[ (4.15) \quad (\exists x) \Box F(x). \]

(4.14) says, In every possible world, something is \(F\). (4.15) says, Something is, in every possible world, \(F\).

If we regard a modal operator as a kind of quantifier over possible worlds, then the de re de dicto distinction corresponds to a permutation of two types of quantifiers. This way of viewing the matter makes the difference readily apparent. For example, (4.15) requires that there be some one thing (at least)—let's call it \(a\)—which is, in every possible world, \(F\). So, \(a\) must be \(F\) in world \(\Gamma\), \(a\) must be \(F\) in world \(\Delta\), and so on. (4.14) requires only that in each world there be something that is \(F\), but it does not require that it be the same thing in each world.

The sort of ambiguity exhibited in (4.11) rarely arises in classical logic, but it can—most famously when the item picked out by a definite description fails to exist. Whitehead and Russell (1925) introduced notation as part of their theory of definite descriptions to mark just such a distinction of scope, similar to the scope indicated by quantifier placement. In Chapter 9 we will formally introduce a related notation we call predicate abstraction that will enable us to mark this distinction explicitly in our modal language. Anticipating a bit, where \(t\) is a singular term (say, “the number of planets”) we will distinguish the de dicto

\[ (4.16) \quad \Box (\lambda x. x \text{ is odd})(t) \]

which says, “It is necessary that \(t\) is odd,” from the de re

\[ (4.17) \quad (\lambda x. \Box (x \text{ is odd}))(t) \]

which says, “\(t\) is such that it is necessary of it that it is odd.”

The de dicto reading (4.16) is false. As we noted, it is a contingent, not a necessary, fact about the universe that there are exactly 9 planets. The expression “the number of planets” picks out a number via one of its contingent properties. If \(\Box\) is to be sensitive to the quality of the truth of a proposition
in its scope, then it must be sensitive as well to differences in the quality of terms designating objects—that is, it will be sensitive as to whether the object is picked out by an essential property or by a contingent one. Since “the number of planets” picks out a number by means of one of its contingent properties, it can pick out different numbers in different possible worlds. In the actual world, that number is 9, but there might have been 10 planets, so in another possible world that number could be 10. The proposition that the number of planets is odd therefore can come out true in some worlds and false in other. It is not necessarily true, and (4.16) is false.

But the de re (4.17) says, “The number of planets is, in every possible world, odd.” This is true. There are exactly 9 planets, and so “the number of planets” designates the number 9. Unlike the de dicto case, the designation of the term has been fixed in the actual world as 9. And that number, in every possible world, is odd.

Predicate abstraction machinery will get a full treatment in Chapter 9. The remarks above must remain informal until then.

Note that there is no de re de dicto distinction for an open sentence like

\( \square (x \text{ is odd}) \).

(4.18) is true of a given object if and only if in every possible world that object is odd. (4.18) is true of the number 9; it is false of the number 10. The mode of specification of the object that is the value of is irrelevant to the truth value of the open sentence; the open sentence is true of if, or not, as the case may be.

Unlike free variables, singular terms—proper names, definite descriptions, function names—require considerable care in modal contexts. To avoid difficulties and confusions, we have chosen to introduce our first-order modal language first without constant or function names, and reserve for later chapters the special problems introduced by them. So, in this chapter we will discuss de re de dicto distinctions of the sort exemplified by (4.12) and (4.13). Distinctions such as those of (4.16) and (4.17) must wait until Chapter 9.

In the next section we will show how the failure to observe the de re de dicto distinction has led to serious errors about the very possibility of coherently doing quantified modal logic.

**Exercises**

**Exercise 4.3.1.** Interpreting \( \square \) as *At all future times*, show informally:

1. \( \square (\forall x) F(x) \) fails to imply \( \forall x \square F(x) \)
2. \( \forall x \square F(x) \) fails to imply \( \square (\forall x) F(x) \)

**Exercise 4.3.2.** Does the sentence “Some things are necessarily F” mask a de re/de dicto distinction? If so, what are the two readings of the sentence?

**4.4. Is Quantified Modal Logic Possible?**

For much of the latter half of the twentieth century, there has been considerable antipathy toward the development of modal logic in certain quarters. Many of the philosophical objectors find their inspiration in the work of W. V. O. Quine, who as early as (Quine, 1943) expressed doubts about the coherence of the project. We will find that one of the main sources of these doubts rests on the failure to carefully observe the distinction between de re and de dicto readings of necessity.

Quantified modal logic makes intelligible the idea that objects themselves, irrespective of how we speak about them, have properties necessarily or contingently. Our semantics for modal logic will not require that any particular object have any particular substantial property either accidentally or essentially, but only that it makes sense to speak this way. We leave it to metaphysicists to fill in their details.

Quine does not believe that quantified modal logic can be done coherently because it takes to be a feature of reality what is actually a feature of language. He says, for example, “Being necessarily or possibly thus and so is in general not a trait of the object concerned, but depends on the manner of referring to the object.” (Quine, 1961a, p. 148) It is instructive to go over Quine’s reasons for holding this view.

Quine (1953) sets up the problem by identifying three interpretations of \( \square \) on which the modality is progressively more deeply involved in our world outlook. On Grade 1, the least problematic level of involvement, \( \square \) is taken to be a metalinguistic predicate that attaches to a name of a sentence, in the same way as the Tarski reading of “is true.” To say that a sentence is necessarily true is no more than to say it is a theorem (of a formal system reasonably close to logic—perhaps including set theory), and the distinction between theorems and nontheorems is clear. But on this reading, there can be no iteration of modal operators, and as a result there is no need for a specifically modal...
logic. There is no interpretation for the many propositional modal logics we studied earlier in this book.

On Grade 2, which is the interpretation we relied on for our discussion of propositional modal logic, $\Box$ is an operator like $\neg$ that attaches to closed formulas, that is, to sentences. But there is an important difference between the logical operators $\Box$ and $\neg$. When $P$ and $Q$ have the same truth value, $\neg P$ and $\neg Q$ also have the same truth value, but $\Box P$ and $\Box Q$ need not have. For example, although the two sentences "9 > 7" and "The number of planets > 7" are both true, only the former is necessarily true. This should not surprise us. Modal logic differs from classical logic in its sensitivity to the quality of a statement's truth, so one would hardly expect $\Box$ to be indifferent to sentences merely happening to have the same truth value.

At Grade 3, $\Box$ is allowed to attach to open formulas, as in $\Box(x > 7)$. This is the level needed to combine modality with quantifiers, for we need to say such things as "Something is such that it is necessarily greater than 7." And it is in the passage from Grade 2 to Grade 3 involvement that Quine finds his problems.\footnote{Of course, Quine has other problems with the connection between necessity and analyticity.}

Quine finds the contrast between the two sentences,

$$9 > 7$$

and

The number of planets > 7

puzzling. The first is necessary, but the second is not. Why is this so? We are speaking about the same thing each time, since the number of the planets is 9. The only difference is in the way in which the thing is picked out. It appears that whether or not the claim is necessary depends, not on the thing talked about, but on the way in which it is talked about. And if so, Quine argues, there can be no clear understanding of whether an open sentence like

$$x > 7$$

is necessarily true or not, for the terms "9" and "the number of planets" on which our intuitions about necessity relied are no longer available.

Here is another example of the phenomenon that has puzzled Quine, this time involving the notion of belief. Although Dr. Jekyll and Mr. Hyde are one and the same person (so the story says), we can very well understand how Robert might believe that Dr. Jekyll is a good citizen and yet not believe that Mr. Hyde is a good citizen. The sentence

Dr. Jekyll is believed by Robert to be a good citizen

is true, but the sentence

Mr. Hyde is believed by Robert to be a good citizen

is false. But then it is a mystery what the open sentence

$x$ is believed by Robert to be a good citizen

is true of. Which individual is this? Dr. Jekyll? Mr. Hyde? Which individual this is appears to depend upon how he is specified, not on the individual himself. The quantified statement "Someone is such that he is believed by Robert to be a good citizen" seems totally uninterpretable.

Grade 2 involvement appears to imply that the behavior of $\Box$ depends on the way things are picked out; Grade 3 involvement requires that $\Box$ be independent of the way things are picked out. Quine argues that we cannot have it both ways. This difference between Grade 2 and Grade 3 involvement looks very much like the distinction between necessity de dicto and necessity de re. But it isn't, and it is important we see that it isn't. If one assumes there is only de dicto necessity at Grade 2, which is surreptitiously what Quine has done, then one will have severe problems with de re necessity, as has happened. If one starts out by assuming that necessity has to do with language and not things, then one will certainly run into problems interpreting modal logic as having do with things and not language.

As we have pointed out on a number of occasions, the complications for names, descriptions and quantifiers indicates no special problem in interpreting open sentences. In particular, it does not show that necessity is more closely connected with how we specify objects than with the objects themselves. One final example using a temporal interpretation always for $\Box$ should make this clear. Contrast the two sentences:

(4.19) The U.S. President will always be a Democrat

(4.20) Bill Clinton will always be a Democrat

Suppose, for simplicity, that Bill Clinton never changes his party affiliation. Understood de re, then, (4.19) and (4.20) are both true. For that man, Bill Clinton or the President of the United States—it does not matter how we specify him—will forever be a Democrat. Understood de dicto, (4.20) is true, but (4.19) is not—for it is highly unlikely that the Democratic party will have a lock on the Presidency forever. Does the difference in truth value show that temporality has more to do with how an object is specified than with the object itself? Hardly. It depends on the fact that the Presidency will be
Changing hands, and Bill Clinton only temporarily holds that office. In the present world, the two coincide; but in later worlds, they won’t.  

EXERCISE 4.4.1. Suppose necessity were intrinsically related to the way in which we pick things out. Then we would have to locate the de re/de dicto distinction in differences in the way designators designate. Discuss how this distinction might be drawn.

4.5. WHAT THE QUANTIFIERS QUANTIFY OVER

In first-order modal logic, we are concerned with the logical interaction of the modal operators ☐ and ◊ with the first-order quantifiers ∀ and ∃. From the perspective of possible world semantics, this is the interaction of two types of quantifiers: quantifiers that range over possible worlds, and quantifiers that range over the objects in those worlds. This interaction leads to complications. In classical logic, Universal Instantiation,

\[(\forall x)\varphi(x) \supset \varphi(x)\]

is valid. But the validity of (4.21) in a modal context depends on which particular possible world semantics we choose, i.e., on what we take our quantifiers to quantify over.

In the language of possible world semantics, the formula

\[(\forall x)\varphi(x) \supset \varphi(x)\]

says that If x is \(\varphi\) in every possible world, then x is \(\varphi\). What is this x? Whatever it is, we will suppose it to be something that occurs in at least one of the possible worlds in our model. And whatever it is, we will suppose that it makes sense to speak of it as existing in more than one possible world, although it need not so exist.

At a very general level, possible world semantics does not require that objects exist in more than one possible world. There is a fairly well-known alternative semantics for first-order modal logic, most vividly put forward by Lewis (1968), which denies that objects can exist in more than one possible world. Lewis takes objects to be “worldbound.” An object in one world, however, can have a counterpart in another possible world. This will be the object in the other world that is (roughly) most similar to the object in this one. (An object will be its own counterpart in any given world.) On the counterpart interpretation, (4.22) says If in every accessible possible world x’s counterpart is \(\varphi\), then x is \(\varphi\). The main technical problem with counterpart theory is that the being-a-counterpart-of relation is, in general, neither symmetric nor transitive, and so no natural logic of equality is forthcoming.

What does it mean for an object to exist in more than one possible world? Here is an example. An assassination attempt was made upon Ronald Reagan’s life. He was hit by the bullet, but only injured. Eventually he recovered from his wounds and continued on as President of the United States for many years. But he was almost killed by that bullet; and he could have been killed. This invites us to consider a counterfactual situation in which that man, Ronald Reagan, was indeed killed by that assassin. We are not considering a situation in which a person just like him was killed by that assassin. We are considering a situation in which that very person, Ronald Reagan, was killed by the assassin. We are imagining the very same person in a counterfactual situation. Since what we mean by a possible world is often just such a counterfactual situation, we are imagining an object—Ronald Reagan—to exist in more than one possible world.

But we have only argued for the coherence of allowing objects to exist in more than one possible world. We have not argued that all objects do, let alone that any of them do. This is actually a choice to be made in setting up modal models: Since a modal model contains many possible worlds, we need to decide whether the domain of discourse should be fixed for the whole of a modal model, or be allowed to vary from world to world, within the model? Taking the domain to be fixed for the whole model provides the simplest formal semantics. We refer to this as constant domain semantics. Allowing modal models to have world-dependent domains gives the greatest flexibility. We refer to this as varying domain semantics. In constant domain semantics, the domain of each possible world is the same as every other; in varying domain semantics, the domains need not coincide, or even overlap. In either case, what we call the domain of the model is the union of the domains of the worlds in the model. (For an interesting application of constant domain modal models in mathematics see (Smullyan and Fitting, 1996), where they provide appropriate machinery for establishing various independence results in set theory.)

We are assuming that free variables have as values objects in the domain

\(^{17}\) There is an alternative reading of a singular term like “the President of the United States” which takes it to refer to an intentional entity. The entity would consist partly of George Washington, partly of Thomas Jefferson, …, that is, of each of the individuals who occupied the office when they did. See (Hughes and Cresswell, 1996) for a discussion of this idea.
of the model, not necessarily in the domain of the world we are in. If we suppose we are dealing with constant domain semantics, then (4.21) holds; more generally,

\[(4.23) \ (\forall x) \varphi(x) \supset \varphi(y)\]

is logically valid. Whenever we speak about an object, that object exists in at least one possible world. In constant domain semantics, what exists at one world exists at all. So any value assigned to \(y\) will be in the range of the quantifier \((\forall x)\). This is analogous to classical logic.

In varying domain semantics, however, the situation is a bit different. No longer does (4.23) hold. Just because everything in this world is \(\varphi\) it doesn’t follow that \(y\) is \(\varphi\), because \(y\) might exist only at worlds other than this one, and so not be in the range of the quantifier \((\forall x)\), which we take as ranging over what exists “here.” This is quite unclassical.

There are, accordingly, two very different ways it could happen that \(x\) is \(\varphi\) at a world \(\Gamma\):

1. \(x\) is \(\varphi\) at \(\Gamma\) and \(x\) is in \(\Gamma\)
2. \(x\) is \(\varphi\) at \(\Gamma\) but \(x\) need not be in \(\Gamma\)

In the classical situation, we speak of something only if it is in the domain of the model, and therefore in the domain of the quantifier. In the modal situation, however, we want to speak about things that do not exist but could, like Pegasus or the golden mountain or a tenth planet in the solar system. This is where we have a choice. One solution is to open up the domain of the actual world to all possible objects and keep the classical quantifier rules like Universal Instantiation intact: this is what has sometimes been called *possibilist* quantification. The other solution is to allow the domain of the actual world to differ from the domain of other possible worlds and abandon classical quantifier rules like Universal Instantiation: this is *actualist* quantification. The possibilist quantifier is evaluated for every element of the model \(M\); the actualist quantifier is evaluated at a world \(\Gamma\) only for elements of the domain of the world \(\Gamma\). Possibilist quantification and actualist quantification correspond to constant domain and varying domain models, respectively. The connection is clear. Constant domain semantics models our intuitions about modality most naturally if we take the domain to consist of possible existents, not just actual ones, for otherwise we would be required to treat every existent as a necessary existent.

**EXERCISE 4.5.1.** Discuss informally how constant domain semantics will differ from varying domain semantics when the modality is given a temporal interpretation. Which is more natural in the temporal reading?

**EXERCISE 4.5.2.** "In constant domain semantics, possible objects exist, so there is no distinction between what is possible and what is actual." Discuss this claim.

**EXERCISE 4.5.3.** "In varying domain semantics, an object can have properties in a world even though it does not exist at that world." Write an essay either defending or criticizing this claim.

### 4.6. Constant Domain Models

We begin our formal treatment of semantics for quantifiers with constant domain models—*possibilist quantification*. In these models, the domain of quantification is the same from world to world. Technically this is somewhat simpler than allowing the domain to vary, and pedagogically it is easier to explain as a first approach. What we present is essentially from (Kripke, 1963b), with some modification of notation. We begin by enhancing the notion of a *frame*, from Definition 1.6.1.

**DEFINITION 4.6.1.** [Augmented Frame] A structure \((\mathcal{G}, \mathcal{R}, \mathcal{D})\) is a constant domain augmented frame if \((\mathcal{G}, \mathcal{R})\) is a frame and \(\mathcal{D}\) is a non-empty set, called the domain of the frame.

The domain of an augmented frame is the set of things over which quantifiers can range, no matter at what world.

To turn a propositional frame into a model, all we had to say was which propositional letters were true at which worlds. The analog of propositional letters now is atomic formulas, and they have a structure that must be taken into account. More specifically, they involve *relation symbols*, which should stand for relations. But since more than one possible world can be involved, we should say which relation each relation symbol represents, at each of the worlds.

**DEFINITION 4.6.2.** [Interpretation] \(I\) is an interpretation in a constant domain augmented frame \((\mathcal{G}, \mathcal{R}, \mathcal{D})\) if \(I\) assigns to each \(n\)-place relation symbol \(R\) and to each possible world \(\Gamma \in \mathcal{G}\), some \(n\)-place relation on the
domain \( D \) of the frame. Thus, \( I(R, \Gamma) \) is an \( n \)-place relation on \( D \), and so each \( n \)-tuple \((d_1, d_2, \ldots, d_n)\) of members of \( D \) either is in the relation \( I(R, \Gamma) \) or is not. If \((d_1, d_2, \ldots, d_n)\) is in the relation \( I(R, \Gamma) \) we will write \((d_1, d_2, \ldots, d_n) \in I(R, \Gamma)\), following the standard mathematical practice of thinking of an \( n \)-place relation as a set of \( n \)-tuples.

**Definition 4.6.3.** [Model] A constant domain first-order model is a structure \( \mathcal{M} = (\mathfrak{g}, R, D, I) \) where \((\mathfrak{g}, R, D)\) is a constant domain augmented frame and \( I \) is an interpretation in it. By the domain of the model \( \mathcal{M} \) we mean the domain of its augmented frame, \( D \). We say \( \mathcal{M} \) is a constant domain first-order model for a modal logic \( L \) if the frame \((\mathfrak{g}, R)\) is an \( L \) frame in the propositional sense.

**Example 4.6.4.** Here is our first example of a constant domain first-order model. Let \( \mathfrak{g} \) consist of three possible worlds, \( \Gamma, \Delta, \) and \( \Omega \), with \( \Gamma R \Delta, \Gamma R \Omega, \) and \( R \) holding in no other cases. Let \( D = \{a, b\} \). Let \( P \) be a one-place relation symbol. It is the only relation symbol we are interested in for now, so we will specify an interpretation only for it, and leave things unspecified for other relation symbols. Now, let \( I(P, \Gamma) \) be the empty set (that is, nothing is in this relation); let \( I(P, \Delta) \) consist of just \( a \); let \( I(P, \Omega) \) consist of just \( b \). This specifies a constant domain first-order model \( \mathcal{M} = (\mathfrak{g}, R, D, I) \). We represent it schematically as follows.

![Diagram](Diagram.png)

We have shown the domain, \{a, b\}, explicitly at each possible world, emphasizing that this is a constant domain model. Notice that we wrote \( P(a) \) at the \( \Delta \) part of the schematic. This is not a formula! It is not, since \( a \) is a member of the domain associated with \( \Delta \), but is not a variable of our formal language. Nonetheless this abuse of notation is handy for specifying models, provided its limitations are understood. Similar remarks apply to \( P(b) \) at \( \Omega \), of course.

Next we must specify truth in constant domain first-order models. Not surprisingly, this is more complicated than it was in the propositional case, and we hinted at the source of the complications in the example above. We would like to have \((\forall x)P(x)\) be true at a possible world, say \( \Gamma \), just in case \( P(x) \) is true at \( \Gamma \) for all members of the domain \( D \). But, we cannot express this by saying that we want \( P(c) \) to be true at \( \Gamma \) for all \( c \in D \), because \( P(c) \) will not generally be a formula of our language. A way around this was introduced many years ago by Tarski for classical logic, and is easily adapted to modal models. For \((\forall x)P(x)\) to be true at \( \Gamma \), we will require that \( P(x) \) be true at \( \Gamma \) no matter what member of \( D \) we might have assigned to \( x \) as its value. But this means we cannot confine ourselves to the truth of sentences at worlds, but instead we must deal with the broader notion of the truth of formulas containing free variables, when values have been assigned to those free variables. This is done using a valuation function—akin to what computer scientists call an environment.

**Definition 4.6.5.** [Valuation] Let \( \mathcal{M} = (\mathfrak{g}, R, D, I) \) be a constant domain first-order model. A valuation in the model \( \mathcal{M} \) is a mapping \( v \) that assigns to each free variable \( x \) some member \( v(x) \) of the domain \( D \) of the model.

What we are about to define is denoted thus:

\[
\mathcal{M}, \Gamma \models_{v} \Phi
\]

where \( \mathcal{M} \) is a constant domain first-order model, \( \Gamma \) is a possible world of the model, \( \Phi \) is a formula, possibly with free variables, and \( v \) is a valuation. Read it as: formula \( \Phi \) is true at world \( \Gamma \) of model \( \mathcal{M} \) with respect to valuation \( v \), where \( v \) tells us what values have been assigned to free variables. (Other books may use different notation.) We need one additional piece of technical terminology, then we can go ahead.

**Definition 4.6.6.** [Variant] Let \( v \) and \( w \) be two valuations. We say \( w \) is an \( x \)-variant of \( v \) if \( v \) and \( w \) agree on all variables except possibly the variable \( x \).

Now, here is the fundamental definition.

**Definition 4.6.7.** [Truth in a Model] Let \( \mathcal{M} = (\mathfrak{g}, R, D, I) \) be a constant domain first-order modal model. For each \( \Gamma \in \mathfrak{g} \) and each valuation \( v \) in \( \mathcal{M} \):

1. If \( R \) is an \( n \)-place relation symbol, \( \mathcal{M}, \Gamma \models_{v} R(x_1, \ldots, x_n) \) provided \((v(x_1), \ldots, v(x_n)) \in I(R, \Gamma)\).
2. \( \mathcal{M}, \Gamma \models_{v} \neg X \iff \mathcal{M}, \Gamma \not\models_{v} X \).
3. \( \mathcal{M}, \Gamma \models_{v} (X \land Y) \iff \mathcal{M}, \Gamma \models_{v} X \) and \( \mathcal{M}, \Gamma \models_{v} Y \).
4. \( \mathcal{M}, \Gamma \models \Box X \iff \) for every \( \Delta \in \mathfrak{I} \), if \( \mathcal{M}, \Delta \models X \).
5. \( \mathcal{M}, \Gamma \models \Diamond X \iff \) for some \( \Delta \in \mathfrak{I} \), \( \mathcal{M}, \mathfrak{I} \Delta \models X \).
6. \( \mathcal{M}, \Gamma \models (\forall x) \Phi \iff \) for every \( x \)-variant \( w \) of \( v \) in \( \mathcal{M}, \mathcal{M}, \mathcal{M}, \Gamma \models \Phi \).
7. \( \mathcal{M}, \Gamma \models (\exists x) \Phi \iff \) for some \( x \)-variant \( w \) of \( v \) in \( \mathcal{M}, \mathcal{M}, \mathcal{M}, \Gamma \models \Phi \).

This definition should be compared with Definition 1.6.3. Propositional connectives other than \( \land \) and \( \lor \) have their usual defined behavior.

The last two items are the key new ones, of course. Item 6 says we should take \( (\forall x) \Phi \) to be true at \( \Gamma \), relative to a valuation \( v \), provided \( \Phi \) is true at every \( \Gamma \) no matter what member of the domain \( \mathfrak{D} \) we assign to \( x \) (keeping the values assigned to other variables unchanged, of course). Likewise 7 says a similar thing about \( (\exists x) \Phi \).

**Proposition 4.6.8.** Suppose \( \mathcal{M} = (\mathfrak{I}, \mathfrak{R}, \mathfrak{D}, \mathfrak{I}) \) is a constant domain model, \( \Gamma \in \mathfrak{I} \), \( v_1 \) and \( v_2 \) are two valuations in \( \mathcal{M} \), and \( \Phi \) is a formula. If \( v_1 \) and \( v_2 \) agree on all the free variables of \( \Phi \), then

\[
\mathcal{M}, \Gamma \models v_1 \Phi \iff \mathcal{M}, \Gamma \models v_2 \Phi
\]

Essentially, this Proposition says that if two valuations agree on the free variables actually present in a formula, the behavior of that formula with respect to the two valuations is the same. The proof is an induction argument on the complexity of \( \Phi \). It is routine, but complicated, and we omit it.

The next Proposition says, roughly, that one variable is as good as another provided we adjust valuations appropriately. In fact, the Proposition below is actually a generalization of the one we just gave. It too has a routine but complicated proof, and it too is omitted.

**Proposition 4.6.9.** Suppose \( \mathcal{M} = (\mathfrak{I}, \mathfrak{R}, \mathfrak{D}, \mathfrak{I}) \) is a constant domain model, \( \Gamma \in \mathfrak{I} \), and \( v_1 \) and \( v_2 \) are valuations in \( \mathcal{M} \). Further suppose that \( \Phi(x) \) is a formula in which \( x \) may have some free occurrences, but \( y \) has no occurrences at all, and \( \Phi(y) \) is the result of replacing all free occurrences of \( x \) with occurrences of \( y \). Finally, suppose \( v_1 \) and \( v_2 \) agree on all the free variables of \( \Phi(x) \) except for \( x \), and \( v_1(x) = v_2(y) \). Then

\[
\mathcal{M}, \Gamma \models v_1 \Phi(x) \iff \mathcal{M}, \Gamma \models v_2 \Phi(y)
\]

Here is the reason this is a generalization of the previous Proposition. Suppose we take for \( x \) a variable that does not actually occur in the formula we are denoting by \( \Phi(x) \). Then the hypothesis that \( v_1 \) and \( v_2 \) agree on all free variables of \( \Phi(x) \) except for \( x \) simply says that they agree on all the free variables present. Thus if we assume that \( x \) has no occurrences in \( \Phi(x) \), the hypothesis of Proposition 4.6.9 becomes the hypothesis of Proposition 4.6.8.

Also, if \( x \) does not occur in \( \Phi(x) \), then \( \Phi(x) \) and \( \Phi(y) \) are the same formula, let us write it more simply as \( \Phi \). So the conclusion of Proposition 4.6.9, in this special case, is also the conclusion of Proposition 4.6.8.

**Definition 4.6.10.** [True at \( \Gamma \)] Let \( \mathcal{M} = (\mathfrak{I}, \mathfrak{R}, \mathfrak{D}, \mathfrak{I}) \) be a constant domain model with \( \Gamma \in \mathfrak{I} \). For a sentence \( \Phi \), if \( \mathcal{M}, \Gamma \models v \Phi \) for some valuation \( v \) in \( \mathcal{M} \) then \( \mathcal{M}, \Gamma \models v \Phi \) for every valuation \( v \) in \( \mathcal{M} \) (by Proposition 4.6.8), and conversely. We abbreviate notation in this situation by writing \( \mathcal{M}, \Gamma \models \Phi \), and we say \( \Phi \) is true at \( \Gamma \).

Now the terminology of Definition 1.8.1 is simply carried over to first-order models. \( \Phi \) is valid in a model if it is true at every world of the model, and so on.

**Example 4.6.11.** We continue Example 4.6.4, and show \( (\forall x) \Diamond P(x) \supset (\forall x) P(x) \) is not valid in the model we gave. All references to \( \mathcal{M} \) are to the model of Example 4.6.4.

Since \( (\forall x) \Diamond P(x) \supset (\forall x) P(x) \) is a sentence, to show

\[
\mathcal{M}, \Gamma \models v (\forall x) \Diamond P(x) \supset (\forall x) P(x)
\]

we must show

\[
\mathcal{M}, \Gamma \not\models v (\forall x) \Diamond P(x) \supset (\forall x) P(x)
\]

where \( v \) is any valuation in \( \mathcal{M} \). We do this as follows.

Let \( w \) be like \( v \) on all variables except that \( w(x) = a \). Now, \( \mathfrak{I}(P, \Delta) \) is the one-place relation that holds of just \( a \), and \( w(x) = a \), so \( \langle w(x) \rangle \in \mathfrak{I}(P, \Delta) \) and by definition we have

\[
\mathcal{M}, \Delta \models w P(x)
\]

and consequently

\[
\mathcal{M}, \Gamma \not\models w P(x).
\]

In the same way we can show that, if \( w' \) is like \( v \) on all variables, except that \( w'(x) = b \), then

\[
\mathcal{M}, \Gamma \not\models w' P(x).
\]

Now \( w \) and \( w' \) are all the \( x \)-variants of \( v \) that there are, since the domain of \( \mathcal{M} \) is \( \{a, b\} \), so by item 6 of Definition 4.6.7,

\[
\mathcal{M}, \Gamma \models v (\forall x) \Diamond P(x).
\]
On the other hand, suppose we had

\[ \mathcal{M}, \Gamma \vdash_v \Diamond (\forall x) P(x). \]

Then we would have one of

\[ \mathcal{M}, \Delta \vdash_v \Diamond (\forall x) P(x) \quad \text{or} \quad \mathcal{M}, \Omega \vdash_v \Diamond (\forall x) P(x). \]

Say we had the first—the argument for the second is similar. Let \( w' \) be the same \( x \)-variant of \( v \) as above—\( w'(x) = b \). By item 6 of Definition 4.6.7 again, we must have

\[ \mathcal{M}, \Delta \vdash_w \Diamond P(x) \]

but we do not, since \( b \) is not in \( \mathcal{M}(P, \Delta) \).

We have shown that

\[ \mathcal{M}, \Gamma \vdash_v \Diamond (\forall x) P(x) \quad \text{and} \quad \mathcal{M}, \Gamma \vdash_v \Diamond (\forall x) P(x) \]

and it follows that

\[ \mathcal{M}, \Gamma \vdash_v \Diamond (\forall x) P(x) \supset \Diamond (\forall x) P(x). \]

Hence the sentence is not valid in the model \( \mathcal{M} \).

**EXAMPLE 4.6.12.** This time we show the converse implication \( \Diamond (\forall x) P(x) \supset (\forall x) \Diamond P(x) \) is valid in all constant domain first-order modal models. Let \( \mathcal{M} = (\mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}) \) be a model, let \( \Gamma \) be an arbitrary member of \( \mathcal{G} \), and let \( v \) be a valuation in \( \mathcal{M} \). To show

\[ \mathcal{M}, \Gamma \vdash_v \Diamond (\forall x) P(x) \supset (\forall x) \Diamond P(x) \]

it is enough to show that

\[ \mathcal{M}, \Gamma \vdash_v \Diamond (\forall x) P(x) \implies \mathcal{M}, \Gamma \vdash_v (\forall x) \Diamond P(x). \]

Suppose

\[ \mathcal{M}, \Gamma \vdash_v \Diamond (\forall x) P(x). \]

Then for some \( \Delta \) with \( \Gamma \mathcal{R} \Delta \),

\[ \mathcal{M}, \Delta \vdash_v (\forall x) P(x), \]

and so if \( w \) is any \( x \)-variant of \( v \) in \( \mathcal{M} \),

\[ \mathcal{M}, \Delta \vdash_w P(x), \]

but \( \Gamma \mathcal{R} \Delta \), and hence

\[ \mathcal{M}, \Gamma \vdash_w \Diamond P(x). \]

Since \( w \) is any \( x \)-variant of \( v \), we have

\[ \mathcal{M}, \Gamma \vdash_v (\forall x) \Diamond P(x). \]

**EXERCISES**

**EXERCISE 4.6.1.** Assume all models are \( K \) models, that is, no special assumptions are made about the accessibility relation. Which of the following sentences are valid in all constant domain first-order modal \( K \) models and which are not?

1. \( [(\exists x) \Diamond P(x) \land \Box (\forall x)(P(x) \supset Q(x))] \supset (\exists x) \Diamond Q(x) \).
2. \( (\forall x) \Box P(x) \supset \Box (\forall x) P(x) \).
3. \( \Box (\forall x) P(x) \supset (\forall x) \Box P(x) \).
4. \( (\exists x) \Diamond P(x) \supset \Box (\exists x) P(x) \).
5. \( \Box (\exists x) P(x) \supset \Box (\exists x) \Box P(x) \).
6. \( (\exists x) \Diamond [\Box P(x) \supset (\forall x) \Box P(x)] \).
7. \( (\exists x) \Diamond [P(x) \supset (\forall x) \Diamond P(x)] \).
8. \( (\exists x) (\forall y) \Box R(x, y) \supset (\forall y) \Box (\exists x) R(x, y) \).

**EXERCISE 4.6.2.** Give a proof of Proposition 4.6.8.

**4.7. VARYING DOMAIN MODELS**

A more general notion of model than that of constant domains allows quantifier domains to vary from world to world. This gives us actualist quantification. Think of the domain associated with a world as what actually exists there, so quantifiers at each world range over the actually existing. Allowing domains to vary complicates the machinery somewhat, but not terribly much. Essentially we just replace the domain set \( \mathcal{D} \) with a domain function that can assign a different domain to each world.

**DEFINITION 4.7.1.** [Augmented Frame] A structure \( (\mathcal{G}, \mathcal{R}, \mathcal{D}) \) is a varying domain augmented frame if \( (\mathcal{G}, \mathcal{R}) \) is a frame and \( \mathcal{D} \) is a function mapping members of \( \mathcal{G} \) to non-empty sets. The function \( \mathcal{D} \) is called a domain function.
As we said above, think of a domain function as associating with each possible world of the frame the set of things that exist at that world, that is, the set over which quantifiers quantify at that world. We often refer to $D(\Gamma)$ as the domain of the world $\Gamma$.

We can think of a constant domain model as a special kind of varying domain model: we simply have a domain function $D$ that assigns the same set to each possible world. Consequently, anything established about varying domain models will apply automatically to constant domain ones. From now on, when appropriate, we will speak indifferently about a constant domain set $D$, or a domain function that is constant, with $D$ as its value at each world. The difference won't matter.

We do face a significant complication before we can properly define varying domain models though. Suppose a formula $\Box (P(x) \lor \neg P(x))$ is true at some possible world $\Gamma$, under a valuation $v$ that assigns $c$ to $x$, where $c$ is something in the domain of world $\Gamma$. If $\Box$ is to have its standard interpretation, then for any world $\Delta$ that is accessible from $\Gamma$, we should have that $P(x) \lor \neg P(x)$ is true at $\Delta$ under the valuation $v$. But, how do we know that $v(x)$, that is, $c$, exists at $\Delta$? More precisely, how do we know $c \in D(\Delta)$? The answer, of course, is that we don't. And so we find ourselves required to consider the truth of a formula at a world when free variables of the formula have values that don't exist at the world in question!

There are two ways out of this problem. One is to allow partial models: take $P(x) \lor \neg P(x)$ to be neither true nor false at $\Delta$ under valuation $v$, when $v(x)$ is not in the domain associated with $\Delta$. The other approach, which is the one we follow, is to say that even though $v(x)$ might not exist in the domain associated with $\Delta$, it does exist under alternative circumstances we are willing to consider, and consequently talk about $v(x)$ is meaningful. Then at $\Delta$, either the property $P$ is true of $v(x)$ or is false of it, and in any event, $P(x) \lor \neg P(x)$ holds. As we said, this is the approach we follow.

**DEFINITION 4.7.2.** [Frame Domain] Let $F = (\mathfrak{g}, R, D)$ be a varying domain augmented frame. The **domain of the frame** is the set $\cup\{D(\Gamma) \mid \Gamma \in \mathfrak{g}\}$. That is, we put together the domains associated with all the possible worlds of the frame. We write $D(F)$ for the domain of the frame $F$.

In a varying domain frame $F = (\mathfrak{g}, R, D)$, if $\Gamma \in \mathfrak{g}$, think of $D(\Gamma)$ as the set of things that actually exist at world $\Gamma$, or in situation or state $\Gamma$. And think of $D(F')$ as the things it makes sense to talk about at $\Gamma$, though these things may or may not exist at $\Gamma$.

Note that if our varying domain frame happens to be constant domain, if $D$ assigns the same domain to each possible world, what we are now calling the domain of the frame turns out to be the domain of the frame as we used the terminology in Section 4.6.

Now, the definition of interpretation is essentially what it was before (Definition 4.6.2), with obvious minor modifications.

**DEFINITION 4.7.3.** [Interpretation] $I$ is an **interpretation** in varying domain augmented frame $F = (\mathfrak{g}, R, D)$ if $I$ assigns, to each $n$-place relation symbol $R$, and to each possible world $\Gamma \in \mathfrak{g}$, some $n$-place relation on the domain $D(F)$ of the frame.

Finally, the definition of model is word for word as it was earlier (Definition 4.6.3), except that we use varying domain augmented frames instead of constant domain ones.

**DEFINITION 4.7.4.** [Model] A varying domain **first-order model** is a structure $M = (\mathfrak{g}, R, D, I)$ where $(\mathfrak{g}, R, D)$ is a varying domain augmented frame and $I$ is an interpretation in it. We say $M$ is a varying domain first-order model for a modal logic $L$ if the frame $(\mathfrak{g}, R)$ is an $L$ frame.

Incidentally, just as with constant domain models, we use the terminology **domain of a model** and mean by it the domain of the underlying frame. We write $D(M)$ for the domain of the model $M$.

**EXAMPLE 4.7.5.** Here is an example of a varying domain first-order model. Let $\mathfrak{g}$ consist of two possible worlds, $\Gamma$ and $\Delta$, with $\Gamma R \Delta$, and $R$ holding in no other cases. Let $D(\Gamma) = \{a\}$, and $D(\Delta) = \{a, b\}$. Let $P$ be a one-place relation symbol. Finally, let $I(P, \Gamma)$ be the empty set (that is, nothing is in this relation), and let $I(P, \Delta)$ consist of just $b$. This specifies a varying domain first-order model $M = (\mathfrak{g}, R, D, I)$. We represent it schematically as follows.

\[
\begin{array}{c}
\Gamma \quad a \\
\downarrow \\
\Delta \quad a, b \quad \vdash \quad P(b)
\end{array}
\]

We return to this example shortly.

To define truth in varying domain models we modify earlier machinery slightly, to ensure that quantifiers really do quantify over things that exist.
DEFINITION 4.7.6. [Valuation] As before, a valuation in a varying domain model \( \mathcal{M} \) is a mapping \( v \) that assigns to each free variable \( x \) some member \( v(x) \) of the domain of the model \( D(\mathcal{M}) \).

DEFINITION 4.7.7. [Variant] The notion of \( x \)-variant is exactly as in Definition 4.6.6. But in addition, we say a valuation \( w \) is an \( x \)-variant of \( v \) at \( \Gamma \) if \( w(x) \) is a member of \( D(\Gamma) \).

DEFINITION 4.7.8. [Truth in a Model] Let \( \mathcal{M} = (\mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}) \) be a varying domain first-order modal model. For each \( \Gamma \in \mathcal{G} \) and each valuation \( v \) in \( D(\mathcal{M}) \):

1-5 Exactly as in Definition 4.6.7.

6. \( \mathcal{M}, \Gamma \models_v (\forall x) \Phi \iff \text{for every } x \)-variant \( w \) of \( v \) at \( \Gamma \), \( \mathcal{M}, \Gamma \models_w \Phi \).

7. \( \mathcal{M}, \Gamma \models_v (\exists x) \Phi \iff \text{for some } x \)-variant \( w \) of \( v \) at \( \Gamma \), \( \mathcal{M}, \Gamma \models_w \Phi \).

The essential difference between the definition above and the earlier Definition 4.6.7 is that, above, we require \( x \)-variants to assign to \( x \) something that is in the domain associated with the possible world we are considering. This, of course, was unnecessary for constant domain models, since all worlds had the same domains.

As before, the behavior of a formula at a world with respect to a valuation is not affected by values the valuation may assign to variables that are not free in the formula. More precisely, Propositions 4.6.8 and 4.6.9 carry over to the varying domain case. Consequently whether a sentence is true or false at a world is completely independent of which valuation we may choose, and so we can suppress mention of valuations when dealing with sentences.

EXAMPLE 4.7.9. We continue with Example 4.7.5, and show the sentence

\( \Diamond (\exists x) P(x) \supset (\exists x) \Diamond P(x) \)

is not valid in the model we gave.

Let \( v \) be any valuation in \( \mathcal{M} \). Let \( w \) be like \( v \) on all variables except that \( w(x) = b \). Now, \( I(P, \Delta) \) is the one-place relation that holds of just \( b \), and \( w(x) = b \), so by definition, we have

\( \mathcal{M}, \Delta \models_w P(x) \).

Since \( b \in D(\Delta) \), \( w \) is an \( x \)-variant of \( v \) at \( \Delta \) and it follows that we have

\( \mathcal{M}, \Delta \models_v (\exists x) P(x) \)

and consequently we have

\( \mathcal{M}, \Gamma \models_v (\exists x) P(x) \).

On the other hand, suppose we had

\( \mathcal{M}, \Gamma \not\models_v (\exists x) \Diamond P(x) \).

Then for some \( x \)-variant \( w \) of \( v \) at \( \Gamma \) we would have

\( \mathcal{M}, \Gamma \not\models_w \Diamond P(x) \).

But \( w(x) \) must be \( a \), since \( a \) is the only member of the domain associated with \( \Gamma \). It follows that

\( \mathcal{M}, \Delta \not\models_w P(x) \)

which can only happen if \( w(x) \) is in the relation \( I(P, \Delta) \), but \( w(x) = a \), and \( a \) is not in this relation. This contradiction shows our supposition was wrong, and so

\( \mathcal{M}, \Gamma \not\models_v (\exists x) \Diamond P(x) \).

Now it follows that

\( \mathcal{M}, \Gamma \not\models_v \Diamond (\exists x) P(x) \supset (\exists x) \Diamond P(x) \)

and so the sentence \( \Diamond (\exists x) P(x) \supset (\exists x) \Diamond P(x) \) is not true at \( \Gamma \), and hence is not valid in the model \( \mathcal{M} \).

EXERCISE 4.7.1. Which of the sentences of Exercise 4.6.1 are valid in all varying domain models and which are not?

4.8. DIFFERENT MEDIA, SAME MESSAGE

We have now seen two radically different semantics: constant domain, and varying domain. Which is, or should be, primary? It turns out, in a very precise sense, that it doesn’t matter. We can formalize the same philosophical ideas either way, with a certain amount of care.

In varying domain semantics we think of the domain of a world as what “actually exists” at that world, in that state, under those circumstances. A quantifier ranges over what actually exists. Thus \( (\exists x) \ldots \) means, there is something, \( x \), that actually exists such that \ldots. The domain of the model
consists of what exists at all worlds, collectively, and so represents what might exist. While quantifiers do not range over the domain of the model, free variables take their values there. Thus a formula \( \Phi(x) \), with \( x \) free, generally has a different behavior from its universal closure, \( (\forall x)\Phi(x) \), in marked contrast to classical logic.

Constant domain semantics, on the other hand, has the same domain for every world. This common domain is analogous to the domain of the model, when varying domain semantics are used. Then, in the constant domain approach, quantifiers range over what does and what might exist. Now \( (\exists x) \ldots \) can be read, "there is something, \( x \), that could exist, such that . . . ."

Suppose, in constant domain semantics, we had an "existence predicate" that tells us which of the things could exist actually exist and which do not. Then we could relativize constant domain quantifiers to this predicate, and achieve an effect rather like being in a varying domain model. Based on the development in (Hughes and Cresswell, 1996), this can be made precise as follows.

**DEFINITION 4.8.1.** [Existence Relativization] Let \( \mathcal{E} \) be a one-place relation symbol. (We intend this to be an existence primitive, and will not use it for any other purpose.) The **existence relativization** of a formula \( \Phi \), denoted \( \Phi^\mathcal{E} \), is defined by the following conditions.

1. If \( A \) is atomic, \( A^\mathcal{E} = A \).
2. \( (\neg x)^\mathcal{E} = (\neg x^\mathcal{E}) \).
3. For a binary connective \( \circ \), \( (X \circ Y)^\mathcal{E} = (X^\mathcal{E} \circ Y^\mathcal{E}) \).
4. \( (\square x)^\mathcal{E} = \square X^\mathcal{E} \).
5. \( (\Diamond x)^\mathcal{E} = \Diamond X^\mathcal{E} \).
6. \( ((\forall x)\Phi)^\mathcal{E} = (\forall x)(\mathcal{E}(x) \circ \Phi^\mathcal{E}) \).
7. \( ((\exists x)\Phi)^\mathcal{E} = (\exists x)(\mathcal{E}(x) \land \Phi^\mathcal{E}) \).

The intuition is that the relativized quantifiers are restricted by a predicate intended to mean: actually exists. Now, the following says to what extent this stratagem succeeds.

**PROPOSITION 4.8.2.** Let \( \Phi \) be a sentence not containing the symbol \( \mathcal{E} \). Then \( \Phi \) is valid in every varying domain model if and only if \( \Phi^\mathcal{E} \) is valid in every constant domain model.

**Proof** The proof of this Proposition amounts to applying formally the motivating ideas, and doing so in a straightforward way.

Part I. Suppose \( \Phi \) is not valid in some varying domain model \( \mathcal{M} = (\mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}) \). We construct a constant domain model in which \( \Phi^\mathcal{E} \) is not valid. This will establish half the Proposition.

Let \( \mathcal{M}' \) be the constant domain model \( (\mathcal{G}', \mathcal{R}', \mathcal{D}', \mathcal{I}') \) constructed as follows. \( \mathcal{G}' \) and \( \mathcal{R}' \) are the same as \( \mathcal{G} \) and \( \mathcal{R} \) respectively. \( \mathcal{D}' \) is the domain of the original model \( \mathcal{M} \). \( \mathcal{I}' \) is like \( \mathcal{I} \) on all relation symbols except \( \mathcal{E} \), and for that, \( \mathcal{I}'(\mathcal{E}(x)) = \mathcal{D}(x) \). Note that since the varying domain model \( \mathcal{M} \) and the constant domain model \( \mathcal{M}' \) have the same model domains, any valuation in one model is a valuation in both.

Now to complete Part I it is enough to show the following. For any formula \( X \) not containing \( \mathcal{E} \) (possibly with free variables, though), for any world \( \Gamma \in \mathcal{G} \), and for any valuation \( \nu \),

\[ \mathcal{M}, \Gamma \models_\nu X \iff \mathcal{M}', \Gamma \models_\nu X^\mathcal{E} \].

This is proved by induction on the complexity of \( X \). The base case, where \( X \) is atomic, is directly by the construction of \( \mathcal{M}' \)—in particular the definition of \( \mathcal{I}' \). The various induction cases are straightforward. We give the existential quantifier case only. Thus, suppose \( X \) is \( (\exists x)Y \), and the result is known for \( Y \).

Suppose first that \( \mathcal{M}, \Gamma \models_\nu (\exists x)Y \). Then for some \( x \)-variant \( w \) of \( \nu \) at \( \Gamma \), \( \mathcal{M}, \Gamma \models_\nu w Y \), so by the induction hypothesis we have \( \mathcal{M}', \Gamma \models_\nu Y^\mathcal{E} \). Since \( w \) is an \( x \)-variant at \( \Gamma \), \( w(x) \in \mathcal{D}(x) \). Then by definition, \( w(x) \in \mathcal{I}'(\mathcal{E}(x)) \), so \( \mathcal{M}', \Gamma \models_\nu \mathcal{E}(x) \land Y^\mathcal{E} \). Thus we have \( \mathcal{M}', \Gamma \models_\nu \mathcal{E}(x) \land Y^\mathcal{E} \), so \( \mathcal{M}', \Gamma \models_\nu ((\exists x)\mathcal{E}(x)) \land Y^\mathcal{E} \).

If we assume \( \mathcal{M}', \Gamma \models_\nu (\exists x)\mathcal{E}(x) \land Y^\mathcal{E} \), it follows by a similar argument that \( \mathcal{M}, \Gamma \models_\nu (\exists x)Y \), and this completes the existential quantifier case for Part I.

Part II. Suppose \( \Phi^\mathcal{E} \) is not valid in some constant domain model. A varying domain model in which \( \Phi \) is not valid must be constructed. We leave this to you as an exercise. ■

Essentially, this Proposition says that instead of working with varying domain models we could work with constant domain models, provided we suitably relativize quantifiers. Relativization provides us an embedding of varying domain semantics into constant domain semantics. We will see below and in Chapter 8 that there is a way of going in the other direction as well, though it is more complicated.

Since each of varying domain and constant domain semantics can simulate the other, semantical machinery does not dictate a solution to us for the problem of what quantifiers must quantify over. Whether we take quantifiers as ranging over the actually existent or over the possibly existent is, in a precise sense, just a manner of speaking. Which way to speak is, finally, a choice to be made based on what seems most natural for what one wants to say.

In this work we treat both varying and constant domain systems—both actualist and possibilist quantification. For some of what we do, it won't
matter which version we choose. When the formal details differ we will say so, and specify which version is appropriate at that point.

**Exercises**

**Exercise 4.8.1.** Let \( \phi \) be the formula

\[
(\exists x)(\square P(x) \supset \square (\forall x) P(x))
\]

where \( P(x) \) is atomic. What is \( \phi^s \)?

**Exercise 4.8.2.** Give the argument for Part II of Proposition 4.8.2.

**4.9. Barcan and Converse Barcan Formulas**

We have seen there is a natural embedding of varying domain modal logic into the constant domain version. Going the other way is more complex, but the route goes through interesting territory. Further, it is territory whose exploration began early in the development of quantified modal logic, and whose connections with the issues of present concern to us were not realized until some time later.

The modal operators \( \square \) and \( \Diamond \) are like disguised quantifiers. To say \( X \) is necessary at a world is to say \( X \) is true at all accessible worlds; to say \( X \) is possible is to say \( X \) is true at some accessible world. The move to first-order modal logic adds the quantifiers \( \forall \) and \( \exists \). We know that in classical first-order logic some quantifier permutations are allowed, but others are not: \( (\forall x)(\forall y) \Phi \) and \( (\forall y)(\forall x) \Phi \) are equivalent; \( (\exists y)(\forall x) \Phi \) implies \( (\forall x)(\exists y) \Phi \); \( (\forall x)(\exists y) \Phi \) does not imply \( (\exists y)(\forall x) \Phi \). These are important facts. A natural question, then, is: what permutations hold between the conventional first-order quantifiers and the modal operators?

Marcus (1946) wrote a study of quantified modal logic—one of the first. Since first-order modal semantics had not yet been invented, all work at the time was axiomatic. It turned out that a particular assumption concerning quantifier/modality permutability was found to be useful. That assumption, or rather, a modernized version of it, has come to be called the Barcan formula. (Barcan was the name Marcus was known by at the time of writing (Marcus, 1946).) Properly speaking, it is not a formula but a scheme.

**Definition 4.9.1.** [Barcan Formula] All formulas of the following forms are Barcan formulas:

1. \( (\forall x)\Box \phi \supset (\forall x)\Box \phi \);
2. \( \Diamond (\exists x) \phi \supset (\exists x) \Diamond \phi \).

There is some (convenient) redundancy in the definition above.

\[ \diamond (\exists x) F \supset (\exists x) \diamond F. \]

is a Barcan formula of form 2. It is equivalent to its contrapositive:

\[ -\diamond (\exists x) F \supset -\diamond (\exists x) F. \]

But \( -\diamond X \equiv \square -X \), and \( -\diamond (\exists x) X \equiv (\forall x) -X \), so this formula in turn is equivalent to

\[ (\forall x)\square -F \supset \square (\forall x) -F, \]

and this is a Barcan formula of form 1. In fact, if we take either form as basic, we get equivalent versions of the other. We simply adopt them both, for convenience.

Over the years it has become common to refer to implications that go the other way as Converse Barcan formulas. It was observed that, for certain natural ways of axiomatizing first-order modal logics, Converse Barcan formulas were provable, though this was not the case for Barcan formulas. Consequently Converse Barcan formulas were thought to be of lesser importance than Barcan formulas, since only assuming the truth of Barcan formulas made a difference. Eventually this point of view proved misleading, and both versions are now seen to play a significant role.

**Definition 4.9.2.** [Converse Barcan Formula] All formulas of the following forms are Converse Barcan formulas:

1. \( \Box (\forall x) \phi \supset (\forall x) \Box \phi \);
2. \( (\exists x) \Diamond \phi \supset \Diamond (\exists x) \phi \).

There is the same redundancy in our definition of Converse Barcan formula that there was for Barcan formulas themselves.

The situation concerning Barcan and Converse Barcan formulas is clarified by possible world semantics. It turns out that these formulas really correspond to fundamental semantic properties of frames, and so the fact that they turned up from time to time over the years was no coincidence. We are about to present the semantical connections, but first a convenient piece of terminology. If we say the Barcan formula is valid (or valid under certain circumstances), we mean all Barcan formulas are; if we say the Barcan formula is not valid, we mean at least one Barcan formula is not. Similarly for the
Converse Barcan formula. It is customary to speak of these formulas as if they were single entities.

The first thing to observe is that neither the Barcan formula nor the Converse Barcan formula is valid generally, that is, in the family of all varying domain $K$ models. We have already seen this for the Barcan formula, in Example 4.7.5, continued in Example 4.7.9. For the Converse Barcan formula, Example 4.9.3 below gives a model in which $(\exists x) \diamond P(x) \supset \diamond (\exists x) P(x)$ fails. We leave it to you to verify this.

**EXAMPLE 4.9.3.** A varying domain counterexample to

$$(\exists x) \diamond P(x) \supset \diamond (\exists x) P(x).$$

$$\Gamma \vdash a, b
\Delta \vdash a \models P(b)$$

Now we introduce a class of frames intermediate between constant domain and varying domain. They turn up frequently in the literature, are very natural mathematically, but do not seem to have much justification from a philosophical point of view.

**DEFINITION 4.9.4.** [Monotonic Frame] The varying domain augmented frame $(\mathfrak{g}, R, D)$ is monotonic provided, for every $\Gamma, \Delta \in \mathfrak{g}$, if $\Gamma \not\models R \Delta$ then $D(\Gamma) \subseteq D(\Delta)$. A model is monotonic if its frame is.

We have already seen some examples. Example 4.6.4, continued in Example 4.6.11, is constant domain, hence monotonic. $(\forall x) \diamond P(x) \supset \diamond (\forall x) P(x)$ is not valid in it. Example 4.7.5, continued in Example 4.7.9, is varying domain and, as it happens, monotonic. It shows that $\diamond (\exists x) P(x) \supset (\exists x) \diamond P(x)$ is not valid in all monotonic models. However, its converse is.

**EXAMPLE 4.9.5.** We show $(\exists x) \diamond P(x) \supset (\exists x) \diamond P(x)$ is valid in all monotonic models. Let $M = (\mathfrak{g}, R, D, I)$ be a monotonic model. We show

$$M, \Gamma \not\models \exists v (\exists x) \diamond P(x) \implies M, \Gamma \not\models \exists v (\exists x) P(x)$$

where $\Gamma \in \mathfrak{g}$ and $v$ is a valuation in $D(M)$.

Assume

$$M, \Gamma \not\models \exists v (\exists x) \diamond P(x),$$

then for some $x$-variant $w$ of $v$ at $\Gamma$,

$$M, \Gamma \not\models w \diamond P(x)$$

and hence for some $\Delta$ such that $\Gamma \not\models R \Delta$,

$$M, \Delta \not\models \diamond w P(x).$$

Now, $\Gamma \not\models R \Delta$ and the model is monotonic, so $D(\Gamma) \subseteq D(\Delta)$. $w$ is an $x$-variant of $v$ at $\Gamma$, so $w(x) \in D(\Gamma)$, and hence $w(x) \in D(\Delta)$. Then $w$ is also an $x$-variant of $v$ at $\Delta$, so we have

$$M, \Delta \not\models \exists v (\exists x) P(x)$$

and hence

$$M, \Gamma \not\models \exists v (\exists x) P(x).$$

It turns out this Example is no coincidence. There is an exact correspondence between the Converse Barcan formula and monotonicity.

**PROPOSITION 4.9.6.** A varying domain augmented frame is monotonic if and only if every model based on it is one in which the Converse Barcan formula is valid.

**Proof** One direction has already been done. If the frame is monotonic the Converse Barcan formula must be valid in every model based on it, as was shown in Example 4.9.5 (in that example, $P(x)$ could have been any formula).

The other direction requires a new argument. Suppose $(\mathfrak{g}, R, D)$ is not monotonic. We produce a model based on it in which one particular Converse Barcan formula is not valid.

Since $(\mathfrak{g}, R, D)$ is not monotonic, there must be $\Gamma, \Delta \in \mathfrak{g}$ with $\Gamma \not\models R \Delta$, but $D(\Gamma) \not\subseteq D(\Delta)$. Say $c \in D(\Gamma)$ but $c \not\in D(\Delta)$. Let $P$ be a one-place relation symbol, and define an interpretation $I$ as follows. $c \in I(P, \Delta)$, but for any $\Omega \in \mathfrak{g}$ other than $\Delta$, $I(P, \Omega)$ is empty. We claim that, in the model $M = (\mathfrak{g}, R, D, I)$,

$$M, \Gamma \not\models (\exists x) \diamond P(x) \supset (\exists x) P(x).$$

Let $v$ be any valuation in $M$, and let $w$ be the $x$-variant of it such that $w(x) = c$. Since $c \in I(P, \Delta),

$$M, \Delta \not\models w P(x);$$

since $\Gamma \not\models R \Delta$,

$$M, \Gamma \not\models w \diamond P(x);$$
and since \( c \in D(\Gamma) \), \( w \) is an \( x \)-variant of \( v \) at \( \Gamma \), so

\[
M, \Gamma \models_v (\exists x) \Diamond P(x).
\]

On the other hand, \((\exists x) P(x)\) is not true at any world. More precisely, if \( \Omega \) is any member of \( \mathcal{G} \),

\[
M, \Omega \not\models_v (\exists x) P(x).
\]

For otherwise, for some \( x \)-variant \( v' \) of \( v \) at \( \Omega \), we would have

\[
M, \Omega \models_{v'} (\exists x) P(x)
\]

and so \( v'(x) \in I(P, \Omega) \). This is not the case if \( \Omega \neq \Delta \) because then \( I(P, \Omega) \) is empty. And this is not the case if \( \Omega = \Delta \) because the only member of \( I(P, \Delta) \) is \( c \), so \( v'(x) \) would have to be \( c \), but \( c \notin D(\Delta) \), so \( v' \) would not be an \( x \)-variant of \( v \) at \( \Delta \).

Now, since \((\exists x) P(x)\) is not true at any world of the model, it is not true at any world accessible from \( \Gamma \), and hence

\[
M, \Gamma \models_v \Diamond (\exists x) P(x)
\]

and this establishes our claim. \( \blacksquare \)

Notice that while monotonicity gets us the Converse Barcan formula, it is still not enough to ensure the Barcan formula itself, as Examples 4.7.5 and 4.7.9 show.

**Definition 4.9.7.** [Anti-Monotonicity] We call a varying domain augmented frame \((\mathcal{G}, \mathcal{R}, D)\) anti-mono\textit{tonic} provided, for \( \Gamma, \Delta \in \mathcal{G} \), \( \Gamma \mathcal{R} \Delta \) implies \( D(\Delta) \subseteq D(\Gamma) \). (Note that the order of inclusion has been reversed.)

**Proposition 4.9.8.** A varying domain augmented frame is anti-monotonic if and only if every model based on it is one in which the Barcan formula is valid.

We leave it to you to prove this, as Exercise 4.9.2.

We still have not brought constant domain frames into the picture. These, too, fit naturally, connected via a somewhat weaker notion.

**Definition 4.9.9.** [Locally Constant Domain] Call a varying domain augmented frame \((\mathcal{G}, \mathcal{R}, D)\) \textit{locally constant domain} provided, for \( \Gamma, \Delta \in \mathcal{G} \), if \( \Gamma \mathcal{R} \Delta \) then \( D(\Gamma) = D(\Delta) \). A model is locally constant domain if its frame is.

In a true constant domain augmented frame, \( D(\Gamma) = D(\Delta) \) for all \( \Gamma, \Delta \), whether or not \( \Gamma \mathcal{R} \Delta \). Certainly every constant domain frame is locally so, but the converse is not true. Now, it follows from the combination of Propositions 4.9.6 and 4.9.8 that having both the Barcan formula and the Converse Barcan formula valid corresponds to the locally constant domain condition. The following puts the final piece in place.

**Proposition 4.9.10.** A sentence \( X \) is valid in all locally constant domain models if and only if \( X \) is valid in all constant domain models.

\textit{Proof} One direction is trivial. If \( X \) is valid in all locally constant domain models, among these are all constant domain models, so it is valid in all of them as well.

In the other direction, suppose \( X \) is \textit{not} valid in some locally constant domain model \( M = (\mathcal{G}, \mathcal{R}, D, I) \); say it is not true at \( \Gamma \). We will produce a constant domain model in which \( X \) is also not valid.

Let us say there is a \textit{path} in the model \( M \) from \( \Delta_1 \) to \( \Delta_2 \) if there is a sequence of worlds, starting with \( \Delta_1 \), finishing with \( \Delta_2 \), with each world in the sequence accessible from the one before it. And let us say a world \( \Delta_2 \) is \textit{relevant to} \( \Delta_1 \) if it is \( \Delta_1 \) itself, or if there is a path from \( \Delta_1 \) to \( \Delta_2 \).

If we are evaluating the truth or falsity of a sentence \( Z \) at a possible world \( \Delta \), we generally must consider the truth or falsity of various subformulas of \( Z \) at certain other worlds, but it is easy to see that \textit{all those worlds will be relevant to} \( \Delta \). Whatever happens at worlds not relevant to \( \Delta \) can have no effect at \( \Delta \) itself.

Now, recall that formula \( X \) was not true at \( \Gamma \in \mathcal{G} \). Let \( \mathcal{G}' \) consist of all members of \( \mathcal{G} \) that are relevant to \( \Gamma \). Let \( \mathcal{R}' \) be \( \mathcal{R} \) restricted to members of \( \mathcal{G}' \), and similarly for \( D' \) and \( I' \). This gives us a new model, \( M' \), which is \( t \) submodel of \( M \). In it the truth or falsity of a formula at \( \Gamma \) is the same as if the original model, since in either model only worlds that are relevant to \( \Gamma \) need be considered when evaluating truth of a formula at \( \Gamma \). Consequently \( X \) is false at \( \Gamma \) in the new model \( M' \).

Finally, \( M' \) must be constant domain. For, suppose \( \Delta \in \mathcal{G}' \). If \( \Delta \) is not \( \Gamma \) itself, there must be a path from \( \Gamma \) to \( \Delta \), and the locally constant domain condition ensures that all worlds along a path will have the same domain hence all worlds in \( M' \) have the same domain as \( \Gamma \). \( \blacksquare \)

Remarkably, as we have seen, simple conditions on frames correspond exactly to the Barcan and the Converse Barcan formulas. Thus these formulas have an importance that goes considerably beyond the technical issue of quantifier permutation. They really say something about the existence assumptions we are making in our semantics. The Converse Barcan formul
says that, as we move to an alternative situation, nothing passes out of existence. The Barcan formula says that, under the same circumstances, nothing comes into existence. The two together say the same things exist no matter what the situation.

Prior (1957) had raised doubts about the Barcan formula. On the temporal reading, it says that if everything that now exists will at all future times be φ, then at all future times everything that then exists will be φ. But this holds generally only if no new things come into existence, and things are always coming into existence. It is important to see that Prior’s quantifier is actually relativized to things that now exist and things that then exist. Relativising the possibilist quantifier turns it into the actualist quantifier. Where there is no relativization, as when we take the quantifier to range over all objects that have been, are, or will be, the Barcan formula will hold.

The very same distinction is operating in the following seeming counterexample to the claim that the Barcan formula holds in every constant domain model. The example is from epistemic logic. Even if

\[
\text{(4.24) Everything is known to be } F.\]

is true, nonetheless, it does not follow that

\[
\text{(4.25) It is known that everything is } F.\]

is also true. The reason is, we might not know that we have everything. We might be able to prove of each number that it is F without being able to prove that every number is F. In fact the quantifier is tacitly relativized in (4.24) to what is known to exist, and this need not be the same as what exists in a world compatible with everything that is known.

In Section 4.8 we saw how to embed varying domain modal logic into the constant domain version. We now have the possibility of going the other way. We briefly sketch how this can be done. We do not give formal details, since it is not an issue that is fundamental here.

We introduced a somewhat complicated notion of logical consequence for propositional modal logic in Section 1.9. That notion extends naturally to a first-order version. Recall, there were really two kinds of deduction assumption: local and global. Based on what was established above, it is not hard to establish that a sentence X is valid in all constant domain models if and only if X is a consequence of the Barcan and the Converse Barcan formulas as global assumptions, in varying domain logic. This allows us to turn questions of validity for constant domain logic into corresponding questions about varying domain logic. Unfortunately, things are not as simple as we would like. Both the Barcan and the Converse Barcan formulas are really schemes, with infinitely many instances. For a given sentence X, which instances should we try working with? Fortunately, we will see that once equality has been introduced, both families of formulas collapse to single instances (Section 8.8. This fact, finally, will give us useful embedding machinery from constant to varying domain logic.

Exercises

EXERCISE 4.9.1. Which of the sentences of Exercise 4.6.1 are valid in all monotonic domain models and which are not? Similarly for anti-monotonic domain models.


EXERCISE 4.9.3. Show the Converse Barcan formula need not be valid in a model whose frame is anti-monotonic.

EXERCISE 4.9.4. Here are four formula schemes:

1. \((\exists x) \Box P(x) \supset \Box (\exists x) P(x)\)
2. \(\Box (\exists x) P(x) \supset (\exists x) \Box P(x)\)
3. \((\forall x) \Diamond P(x) \supset \Diamond (\forall x) P(x)\)
4. \(\Diamond (\forall x) P(x) \supset (\forall x) \Diamond P(x)\)

Just as we gave two versions of the Barcan formula, and observed they came in pairs, the same is true for the schemes above. Determine which pairs of schemes constitute equivalent assumptions.

EXERCISE 4.9.5. For the formula schemes in Exercise 4.9.4, determine the status of their validity assuming: constant domains; varying domains; monotonic domains; anti-monotonic domains.